TOTAL INCREMENT THEOREM

INCREMENTS

The *increment* δx of a variable x is the change in x as it increases or decreases from one value $x = x_0$ to another value $x = x_1$. Here, $\delta x = x_1 - x_0$ and we may write $x_1 = x_0 + \delta x$.

If the variable x is given an increment δx from $x = x_0$, (i.e., if x changes from $x = x_0$ to $x = x_0 + \delta x$) then the function $y = f(x)$ is incremented from $y = f(x_0)$ by $\delta y = f(x_0 + \delta x) - f(x_0)$. The ratio

$$\frac{\delta y}{\delta x} = \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$

is called the *average rate of change* of the function in the interval between $x = x_0$ and $x = x_0 + \delta x$

THE DERIVATIVE

The *derivative* of a function $y = f(x)$ with respect to x at the point $x = x_0$ is defined as

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}, \text{ provided this limit exists}$$

The derivative of $y = f(x)$ can have any of the following notations: $\frac{d}{dx} y$, $\frac{dy}{dx}$, y', $\dot{y}$, $f'(x)$ or $\frac{d}{dx} f(x)$

TOTAL INCREMENT THEOREM

A function of a single variable may be expanded about a point using Taylor's theorem to give the series

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \frac{(x - x_0)^3}{3!} f'''(x_0) + \cdots$$
$$\cdots + \frac{(x - x_0)^{n-1}}{(n-1)!} f^{(n-1)}(x_0) + R_n$$

$f'(x_0)$, $f''(x_0)$, $\cdots$ etc are derivatives of the function $f(x)$ evaluated at $x = x_0$. R_n is the remainder after n terms and

$$R_n = \frac{(x - x_0)^n}{n!} f^{(n)}(\xi) \qquad x_0 \le \xi \le x$$

Using the increment $\delta x = x - x_0$ the Taylor series can be written as

$$f(x) = f(x_0) + \delta x\, f'(x_0) + \frac{(\delta x)^2}{2!} f''(x_0) + \frac{(\delta x)^3}{3!} f'''(x_0) + \cdots$$

If δx is small then $(\delta x)^2$, $(\delta x)^3$, ... are of second order and small and we may write as an approximation (the linearisation of f)

$$f(x) \simeq f(x_0) + \delta x\, f'(x_0)$$

Also, if we write $y = y(x)$ then $y_0 = y(x_0)$ and we have an expression for the increment $\delta y = y(x) - y(x_0) = y - y_0$ as

$$\delta y \simeq y'(x_0)\,\delta x$$

For a function of two variables, say $z = z(x, y)$, the Taylor series expansion of z about $x = x_0$, $y = y_0$ is

$$z = z(x_0, y_0) + (x - x_0)\left.\frac{\partial z}{\partial x}\right|_0 + (y - y_0)\left.\frac{\partial z}{\partial y}\right|_0$$
$$+\frac{1}{2!}\left\{(x - x_0)^2 \left.\frac{\partial^2 z}{\partial x^2}\right|_0 + (y - y_0)^2 \left.\frac{\partial^2 z}{\partial y^2}\right|_0 + (x - x_0)(y - y_0)\left.\frac{\partial f}{\partial x}\right|_0 \left.\frac{\partial f}{\partial y}\right|_0\right\} + \cdots$$

where $z(x_0, y_0)$ is the function z evaluated at $x = x_0$ and $y = y_0$, and $\left.\frac{\partial z}{\partial x}\right|_0$, $\left.\frac{\partial z}{\partial y}\right|_0$, $\left.\frac{\partial^2 z}{\partial x^2}\right|_0$, etc are partial derivatives of the function z evaluated at $x = x_0$ and $y = y_0$.

Again, using the increments $\delta x = x - x_0$, $\delta y = y - y_0$ and $\delta z = z - z_0$, and ignoring terms involving products we may write that for $z = z(x,y)$ then

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

We may extend this to functions of three or and more variables and write the *Total Increment Theorem* as

For a function $w = w(x,y,z,\ldots t)$ the *total increment* δw is

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z + \cdots + \frac{\partial w}{\partial t}\delta t$$

A geometric interpretation of this approximation for two independent variables is given by the area of a rectangle $WXYZ$ whose sides are h and w and area is $A = h \times w$

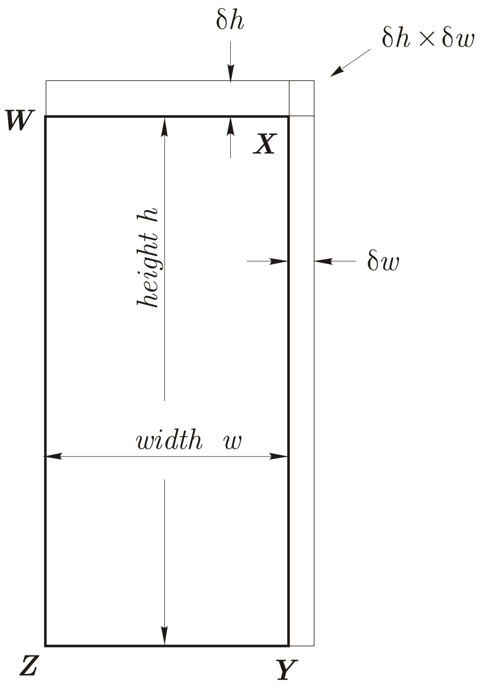

Figure 1

Remembering that by definition $A = A_0 + \delta A$ and $\delta A = A - A_0$ where $A_0 = hw$ and $A = (h + \delta h)(w + \delta w)$, the increment δA is

$$\begin{aligned}\delta A &= (h+\delta h)(w+\delta w) - hw \\ &= hw + h\,\delta w + w\,\delta h + \delta h\,\delta w - hw \\ &= h\,\delta w + w\,\delta h + \delta h\,\delta w\end{aligned}$$

The Total Increment Theorem gives δA as

$$\begin{aligned}\delta A &\simeq \frac{\partial A}{\partial h}\delta h + \frac{\partial A}{\partial w}\delta w \\ &= w\,\delta h + h\,\delta w\end{aligned}$$

We can see here that the increment δA given by the Total Increment Theorem differs from the true δA by the value $\delta h\,\delta w$, which is the small area in the top-right corner of Figure 1. If the increments δh and δy are small then the product $\delta h\,\delta w$

will be very much smaller and the Total Increment Theorem may give a good approximation of the increment δA.

SOME SIMPLE SURVEYING APPLICATIONS OF THE TOTAL INCREMENT THEOREM

In surveying applications, measured quantities can be affected by systematic errors or constant errors. Systematic errors are those that accord to some law or rule, say for example, the correction of EDM distances for atmospheric conditions (temperature and pressure). Atmospheric corrections will vary according to the conditions and if they are not made then EDM distances will be in error. Constant errors are those whose sign and magnitude do not vary. Again, using EDM as an example, if the instrument index error is not allowed for (i.e., the measurement is not corrected for the known error) EDM distances will be affected by a constant amount. The effects of systematic and constant errors in surveying measurements can be determined by using the Total Increment Theorem.

Example 1

Say a bearing ϕ and distance s is measured from a fixed point A to a point P to determine its coordinates. The coordinates of P are given by

$$E_P = E_A + s\sin\phi$$
$$N_P = N_A + s\cos\phi$$

and we say that E_P and N_P are functions of the independent variables s and ϕ

$$E_P = E_P(s,\phi)$$
$$N_P = N_P(s,\phi)$$

If the measurements s and ϕ are affected by systematic or constant errors δs and $\delta\phi$ then the "error" in the coordinates E_P and N_P induced by δs and $\delta\phi$ are δE_P and δN_P given by the Total Increment Theorem as

$$
\begin{aligned}
\delta E_P &\simeq \frac{\partial E_P}{\partial s}\delta s + \frac{\partial E_P}{\partial \phi}\delta\phi \\
&\simeq \sin\phi\,(\delta s) + s\cos\phi\,(\delta\phi) \\
\delta N_P &\simeq \frac{\partial N_P}{\partial s}\delta s + \frac{\partial N_P}{\partial \phi}\delta\phi \\
&\simeq \cos\phi\,(\delta s) - s\sin\phi\,(\delta\phi)
\end{aligned}
$$

Note here that the increments (or errors) δE_P and δN_P are the sums of separate increment terms, one of which is negative. In most applications of the Total Increment Theorem we would wish to estimate the <u>maximum values</u> given errors that don't exceed certain limits, say $\pm\delta s$ and $\pm\delta\phi$; and in such cases we would write

$$
\begin{aligned}
\delta E_{\max} &\simeq \left|\sin\phi\,(\delta s)\right| + \left|s\cos\phi\,(\delta\phi)\right| \\
\delta N_{\max} &\simeq \left|\cos\phi\,(\delta s)\right| + \left|s\sin\phi\,(\delta\phi)\right|
\end{aligned}
$$

Say the measured distance is $s = 165.30 \pm 0.01$ m and the measured bearing is $\phi = 126^\circ\, 31'\, 00'' \pm 10''$, i.e., we are saying that the constant or systematic errors do not exceed $\delta s = 0.01$ m or $\delta\phi = 10'' = 4.8481E-05$ radians

$$
\begin{aligned}
\delta E_{\max} &\simeq |0.0080| + |-0.0048| = 0.0128 \text{ m} \\
\delta N_{\max} &\simeq |-0.0060| + |0.0064| = 0.0124 \text{ m}
\end{aligned}
$$

We could verify this result by computation using the values $s_{\max} = 165.31$, $s_{\min} = 165.29$ and $\phi_{\max} = 126^\circ\, 31'\, 10''$, $\phi_{\min} = 126^\circ\, 30'\, 50''$ in all combinations for the change in east and north coordinates ΔE, ΔN

$$
\Delta E = \begin{cases}
s_{\max}\sin\phi_{\max} & = 132.8522 \text{ m} \\
s_{\max}\sin\phi_{\min} & = 132.8617 \text{ m} \\
s_{\min}\sin\phi_{\max} & = 132.8361 \text{ m} \\
s_{\min}\sin\phi_{\min} & = 132.8457 \text{ m}
\end{cases}
$$

$$
\Delta N = \begin{cases}
s_{\max}\cos\phi_{\max} & = -98.3752 \text{ m} \\
s_{\max}\cos\phi_{\min} & = -98.3624 \text{ m} \\
s_{\min}\cos\phi_{\max} & = -98.3633 \text{ m} \\
s_{\min}\cos\phi_{\min} & = -98.3505 \text{ m}
\end{cases}
$$

The range in ΔE is 0.0256 m and in ΔN is 0.0247 m, and they can be interpreted as twice the increments $\delta E_{\max}$ and $\delta N_{\max}$, which by inspection, we can see to be correct. This is a practical verification of a very useful tool for the estimation of the effects of errors on computed quantities.

The Total Increment Theorem can also be used to assess the need for rigorously computed or observed quantities in certain formula.

Example 2

Say that I am observing a radiation to occupation and I wish to compute the offset to a traverse line. The distance from the instrument to the occupation is $s = 10$ m and the angle between the traverse and the radiation is $\theta = 30^\circ$. The offset $x = s\sin\theta$. Now assume that the distance s is exact, i.e., there is no error in the distance and the offset x is a function of the angle θ only. Let's say that I wish to be able to compute the offset correct to 0.005 m; how accurately do I need to measure the angle?

We can use the Total Increment Theorem as follows:

$$x = s\sin\theta$$

then

$$\delta x \simeq \sin\theta\,(\delta s) + s\cos\theta\,(\delta\theta)$$

but s is known without error, hence $\delta s = 0$ and

$$\delta x \simeq s\cos\theta\,(\delta\theta)$$

This equation can be re-arranged as

$$\delta\theta \simeq \frac{\delta x}{s\cos\theta} = \frac{0.005}{10\cos 30^\circ} = 5.774E-04 \text{ radians} \simeq 119''$$

So, we can say that we only need to observe the angle to an accuracy of $\pm 02'$ for a computational accuracy of 0.005 m in the offset.

Example 3

In Map Grid Australia (MGA) computations, east and north coordinates are computed using *plane distances L.* In the field we measure slope distances and reduce them to horizontal distances H on a local plane and these local plane distances H are assumed to be at a mean height h_m above the reference ellipsoid. Next, the local plane distances H are reduced to chord-distances c of the ellipsoid by a height scale factor. c is the chord of an arc-distance s on the ellipsoid, and for short lines (<10 km) the ellipsoid can be approximated by a sphere of radius R_α and the difference between the arc-distance s and chord-distance c is negligible. So for short lines (<10 km) we have

$$s = \left(\frac{R_\alpha}{R_\alpha + h_m} \right) H$$

where $\left(\frac{R_\alpha}{R_\alpha + h_m} \right)$ is Height Scale Factor and R_α is the radius of curvature of the reference ellipsoid in the direction of the line.

Arc-distances s are converted into plane distances L by applying a Line Scale Factor K. This process is given by

$$L = Ks = K \left(\frac{R_\alpha}{R_\alpha + h_m} \right) H = \text{Line Scale Factor} \times \text{Height Scale Factor} \times H$$

The product of the two scale factors is known as the *Combined Scale Factor*

Now, how accurately do we need to know R_α when computing plane distances L given the local plane distance H, the line scale factor K and the mean height of the local plane h_m?

It is known that the mean radius of curvature of the reference ellipsoid has maximum and minimum values in Victoria of approximately 6378700 m and 6371400 m. This is a variation of 7300 m (7.3 km). Let's say we choose a value of $R_\alpha = 6372000$ m and we will accept that $\delta R_\alpha = 20000$ m (i.e., a 20 km error in R_α).

The Total Increment Theorem gives

$$\delta L \simeq \left\{ \frac{h_m}{\left(R_\alpha + h_m\right)^2}\,\delta R_\alpha - \frac{R_\alpha}{\left(R_\alpha + h_m\right)^2}\,\delta h_m \right\}(K \times H)$$

Let's also assume that the mean height $h_m = 500$ m is known without error (i.e., $\delta h_m = 0$), the local plane horizontal distance $H = 1000$ m and the Line Scale Factor $K = 1.000000$. Our equation becomes

$$\delta L \simeq \left\{ \frac{h_m}{\left(R_\alpha + h_m\right)^2}\,\delta R_\alpha \right\}(K \times H)$$

Substituting in the values for h_m, R_α, δR_α, K and H gives $\delta L = 0.000246$ m. So we may conclude that a value of $R_\alpha = 6372000$ m is suitable for use anywhere in Victoria.

It should be noted that the range of L for an error δL is $L_{\max} - L_{\min}$ where $L_{\max} = L + \delta L$ and $L_{\min} = L - \delta L$ giving the range $= 2(\delta L) = 0.000492$ m.
This can be verified by computation where

$$L_{\max} = \left(\frac{R_\alpha + \delta R_\alpha}{R_\alpha + \delta R_\alpha + h_m} \right) KH = 999.921783 \text{ m}$$
$$L_{\max} = \left(\frac{R_\alpha - \delta R_\alpha}{R_\alpha - \delta R_\alpha + h_m} \right) KH = 999.921291 \text{ m}$$

and

$$\text{range} = 999.921783 - 999.921291 = 0.000492 = 2(\delta L)$$